ON POTENTIALS OF POSITIVE MASS*

PART I

BY

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I. Introduction

A recent contribution to potential theory is characterized by the names of Lebesgue, Wiener, Kellogg, Vasilesco and Bouligand. Central features of this contribution are the notions of capacity and of regular boundary point, which are related to each other by Kellogg's hypothetical lemma.† A recent memoir by de la Vallée Poussin reinterprets these theories in the light of potentials of positive mass and the Poincaré sweeping out process.‡ In the present memoir, the author's aim is to push on the development of these central problems of mass distribution, regularity, capacity, and approximation, and to answer definitely some of the questions which have become important. Fortunately there is already at hand, in the analysis of the general integral or linear functional of Radon, Daniell, and F. Riesz, the precise mathematical tool which is necessary.§

1. **Integrals and potentials.** Let F be a closed bounded point set, T the infinite domain lying in the complement of F, whose boundary t consists entirely of points of F. We consider an arbitrary distribution of positive mass $f(e)$ on F, finite in total amount.‖ The potential of this mass, at a point M, is given by the generalized Stieltjes integral

$$U(M) = \int \frac{1}{MP}\, df(e_P)$$

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† O. D. Kellogg, *Foundations of Potential Theory*, Berlin, 1929, p. 337.

‡ C. de la Vallée Poussin, *Extension de la méthode du balayage de Poincaré, et problème de Dirichlet*, Annales de l'Institut Henri Poincaré, vol. 2 (1932), pp. 169–232.

§ G. Radon, *Theorie und Anwendungen der absolut additiven Mengenfunktionen*, Sitzungsberichte der Akademie der Wissenschaften in Wien, vol. 122, IIa (1913), pp. 1295–1438; *Über die Randwertaufgaben beim logarithmischen Potential*, ibid., vol. 128 (1919), pp. 1123–1167.

P. J. Daniell, (1) *A general form of integral*, Annals of Mathematics, vol. 19 (1918), pp. 279–294; (2) *Further properties of the general integral*, ibid., vol. 21 (1920), pp. 203–220.

F. Riesz, *Über lineare Funktionalgleichungen*, Acta Mathematica, vol. 41 (1916–18), pp. 71–98.

‖ That is, $f(e)$ is an additive, bounded, not negative function of Borel measurable point sets e, such that $f(e \cdot CF) = 0$, CF being the complement of F.

extended over the whole of space; when necessary we denote this entire space by W, so that $W-F$, $W-t$, etc. have meaning.

Let s be a closed bounded set which is the boundary of a domain Σ; as a whole or in part s may be the boundary also of other domains $B_1, B_2, \cdots$, which thus constitute in their totality an open set B. The set $s+B$ is a closed set G which may or may not have points in common with F.

It is of frequent application that if $f(e)$ is a distribution of positive mass on F, bounded in total amount, the same is true of the function

$$f'(e) = f(e \cdot E)$$

where E is any fixed set, measurable Borel. Also if $f'(e)$, $f''(e)$ are two such functions, with $f(e) = f'(e) + f''(e)$, and $\phi(P)$ a continuous point function,

$$\int_W \phi(P) df(e) = \int_W \phi(P) df_1'(e) + \int_W \phi(P) df''(e).$$

The same equation remains valid for the generalized integral, $\phi(P)$ not being continuous, as far as the integrals exist.

Let now $f_1(e), f_2(e), \cdots$ form a denumerable sequence of such functions chosen so that $f_1(F) + f_2(F) + \cdots$ is convergent. The functions

$$f^n(e) = f_1(e) + f_2(e) + \cdots + f_n(e),\ f(e) = f_1(e) + f_2(e) + \cdots,$$
$$r^n(e) = f_{n+1}(e) + f_{n+2}(e) + \cdots$$

are distributions of positive mass on F, finite in total amount, and if $\phi(P)$ is continuous,

$$\int_W \phi(P) df(e) = \lim_{n=\infty} \int_W \phi(P) df^n(e). \tag{1}$$

In fact,

$$\int_W \phi(P) df(e) = \int_W \phi(P) df^n(e) + \int_W \phi(P) dr^n(e),$$

and, as in (3), below,

$$\left| \int_W \phi(P) dr^n(e) \right| \leqq r^n(F)(\text{u.b. on } F \text{ of } |\phi(P)|).$$

We introduce the function

$$h^N(M, P) = 1/(MP), \qquad MP \geqq 1/N,$$
$$= N, \qquad MP < 1/N.$$

This is a continuous function of P to which (1) applies. Hence, for all N,

$$\int_W h^N(M, P)df(e_P) = \lim_{n=\infty} \int_W h^N(M, P)df^n(e_P) \leqq \liminf_{n=\infty} \int_W \frac{1}{MP} df^n(e_P),$$

and

$$\int_W \frac{1}{MP} df(e_P) \leqq \liminf_{n=\infty} \int_W \frac{1}{MP} df^n(e_P),$$

admitting $+\infty$ as a possible value of the integral. Also

$$\int_W h^N(M, P)df(e_P) \geqq \int_W h^N(M, P)df^n(e_P) \text{ and}$$

$$\int_W \frac{1}{MP} df(e_P) \geqq \int_W \frac{1}{MP} df^n(e_P)$$

for every n. Hence

$$\int_W \frac{1}{MP} df(e_P) \geqq \limsup_{n=\infty} \int_W \frac{1}{MP} df^n(e_P).$$

From the two inequalities we deduce

$$\int_W \frac{1}{MP} df(e_P) = \lim_{n=\infty} \int_W \frac{1}{MP} df^n(e_P), \tag{2}$$

which is an identity in the potential functions of the various masses.

Let E be a set measurable Borel. If $\phi(P)$ is *bounded and continuous in* W and $|\phi(P)| \leqq N$ on E, we have

$$\left| \int_W \phi(P)df(E\cdot e_P) \right| \leqq Nf(E), \tag{3}$$

and if $\phi_1(P)$ and $\phi_2(P)$ are *bounded and continuous in* W *and identical on* E, then

$$\int_W \phi_1(P)df(E\cdot e_P) = \int_W \phi_2(P)df(E\cdot e_P). \tag{4}$$

So much is seen, for (3), by taking the Stieltjes integral as the limit of a Riemann sum on a net; and (4) is an immediate consequence of (3).

But further, *if the set E is merely measurable Borel, and ϕ, ϕ_1, ϕ_2 are merely measurable Borel in W, then* (3) *is valid if* $|\phi(P)| \leqq N$ *on E, and* (4) *if* $\phi_1(P) = \phi_2(P)$ *on E, provided the integrals are convergent.*

In fact, if we define the class T_0, of Daniell, as the class of functions $\phi(P)$ on E which can be extended so as to be bounded and continuous in W, and take the range of the variable P as the set E, the integral defined by the equation

$$I = \int_E \phi(P)df(e) = \int_W \phi(P)df(E\cdot e_P)$$

is uniquely determined by the values of $\phi(P)$ on E, satisfies the postulates (C), (A), (L), (P) of Daniell,* and is therefore a general I-integral on E. It is merely the postulate (L) which requires attention:

(L) If $\phi_1(P) \geqq \phi_2(P) \geqq \cdots$ on E and $\lim (n=\infty)\phi_n(P)=0$ for all P in E, then $\lim (n=\infty)\ I(\phi_n)=0$.

If E is closed, the $\phi_n(P)$ approaches 0 uniformly on E, and (3), for continuous functions, yields the conclusion (L). If E is not closed, and $\phi_1(P) \leqq N_1$ on E, the lesser, nevertheless, of the two values $\phi_1(P)$, N_1 forms a continuous function $\psi(P)$ on W. Moreover $I(\psi)=I(\phi_1)$ by (4). Hence without loss of generality we may suppose $\phi_n(P) \leqq N_1$ in W. Now, given $\epsilon>0$, E contains a closed set E_1, such that $f(E)-f(E_1)<\epsilon/N_1$; moreover $f(e\cdot[E-E_1])$ is a positive distribution of mass. We have

$$\begin{aligned} I(\phi_n) - I_1(\phi_n) &= \int \phi_n(P)df(E\cdot e_P) - \int \phi_n(P)df(E_1\cdot e_P) \\ &= \int \phi_n(P)df([E - E_1]\cdot e_P) < N_1\epsilon/N_1 = \epsilon. \end{aligned}$$

But $\lim I_1(\phi_n)=0$; whence $\lim I(\phi_n)\leqq\epsilon$. Consequently also $\lim I(\phi_n)=0$.

Let Γ_ρ be a spherical region of radius ρ and center Q, and denote by $U_{\bar{\Gamma}_\rho}(Q)$ the contribution to the potential at Q due to the mass on Γ_ρ. We note, with de la Vallée Poussin, that *if $U(Q)$ is finite,*

$$\lim_{\rho=0} U_{\Gamma_\rho}(Q) = 0. \tag{5}$$

In fact, given $\epsilon>0$, we can choose N so that

$$U^{(N)}(Q) = \int_W h^N(Q, P)df(e_P) > U(Q) - \epsilon/2.$$

* Daniell, loc. cit. (1), p. 280. The postulates (C), (A), (P) are as follows: (C) If $\theta(P)=c\phi(P)$ on E, $I(\theta)=cI(\phi)$; (A) If $\theta(P)=\phi_1(P)+\phi_2(P)$ on E, $I(\theta)=I(\phi_1)+I(\phi_2)$; (P) If $\phi(P)\geqq0$ on E, $I(\phi)\geqq0$. If these postulates are satisfied for functions of the class T_0, which has to be closed with respect to the operations of addition, multiplication by a constant, logical addition and logical multiplication, they remain as properties for the class of functions to which the operation I is generalized. This class in our case contains all those functions on E which arise from functions bounded and measurable Borel on W.

Hence, with this value of N, since the integrands are positive,

$$\int h^N(Q, P)df(\Gamma_\rho \cdot e_P) > U_{\Gamma_\rho}(Q) - \epsilon/2, \text{ for all } \rho.$$

But $h^N(Q, P)$ is bounded and lim $(\rho=0)f(\Gamma_\rho)=0$, since otherwise $U(Q)$ would be $+\infty$; and we can therefore choose $\rho_0>0$ so that

$$\int h^N(Q, P)df(\Gamma_\rho \cdot e_P) < \epsilon/2, \qquad \rho \leqq \rho_0.$$

Whence

$$U_{\Gamma_\rho}(Q) < \epsilon, \qquad \rho \leqq \rho_0.$$

2. **Potentials and superharmonic functions.** A function $u(M)$ is superharmonic in a bounded domain Ω if (i) it is lower semicontinuous in Ω, not identically equal to $+\infty$, and (ii) given any regular closed surface σ, contained with its interior in Ω, and any function $v(M)$ continuous within and on σ, harmonic within σ, such that $v(M) \leqq u(M)$ on σ, then also $v(M) \leqq u(M)$ within σ.† For (ii) may be substituted the statement that $u(M)$ is at least as great as its mean on any spherical surface of center M and radius ρ, for all ρ sufficiently small, depending on M. Instead of a spherical surface, a spherical volume may be used.

It will be convenient to speak of a function simply as superharmonic if it is superharmonic in every bounded domain, and as superharmonic in an infinite domain if it is superharmonic in every bounded domain contained in the infinite domain.

Let Ω_1 be any domain contained with its boundary Ω_1^* in Ω. A function which is superharmonic in Ω cannot take on its lower bound b for Ω_1 at any point of Ω_1 unless it is identically constant. On the other hand, given $\epsilon>0$, the set of points in $\Omega_1+\Omega_1^*$ where $u(M) \leqq b+\epsilon$ is closed and not empty. Hence $u(M)$ must take on the value b at some point of Ω_1^*. It is F. Riesz's fundamental theorem that $u(M)$ may be written in Ω_1 as the sum of a potential of a distribution of positive mass on Ω_1, finite in total amount, and a function harmonic in Ω_1.‡

The potential $U(M)$ satisfies the conditions (i), (ii), and accordingly is superharmonic. Conversely, *if $u(M)$ satisfies the conditions* (i), (ii) *in W, is harmonic outside F and vanishes continuously at ∞, it must be the potential*

† For the analysis of superharmonic functions, see F. Riesz, (1) *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel*, Acta Mathematica, vol. 48 (1926), pp. 329–343; (2) *same title*, ibid., vol. 54 (1930), pp. 321–360.

‡ A proof of F. Riesz's theorem is given in §4, below.

$U(M)$ of a distribution of positive mass on F, finite in total amount.†

In order to prove this statement we apply the theorem of F. Riesz to a spherical region Ω_1, of radius ρ and center at a point of F, which contains F in its interior. Since $u(M)$ is harmonic outside F the distribution of positive mass must lie on F. Given $\epsilon>0$, we take ρ so large that the potential $U(M)$ of this distribution is $\leqq\epsilon/2$ on Ω_1^*, and also so large that $|u(M)|$ is $\leqq\epsilon/2$ on Ω_1^*, and write, for all M in W,

$$u(M) = U(M) + v(M).$$

Then $v(M)$ is harmonic in Ω_1 and continuous in $\Omega_1+\Omega_1^*$, and $|v(M)|\leqq\epsilon$ on Ω_1^*. Hence $|v(M)|\leqq\epsilon$ in Ω_1. But this means that $v(M)\equiv 0$.

LEMMA. *If $U_1(M)$ and $U_2(M)$ are potentials of positive mass distributions $f_1(e)$, $f_2(e)$ on F, and $U_2(M)\geqq U_1(M)$, for all M, then $f_2(F)\geqq f_1(F)$.*

In fact, if $U_2(M)\equiv U_1(M)$ in T their total masses are equal:

$$\int_C \frac{\partial U_1}{\partial n}\,d\sigma = \int_C \frac{\partial U_2}{\partial n}\,d\sigma,$$

C being a smooth surface sufficiently large to contain F in its interior.‡ If $U_2(M_1)>U_1(M_1)$ for some M_1 in T, it follows that $U_2(M)>U_1(M)$ for all M in T, for $U_2(M)\geqq U_1(M)$ and both functions are harmonic in T. Hence by taking C as a spherical surface of radius ρ sufficiently large and developing U_2-U_1 about ∞, we see that

$$\int_C \frac{\partial U_2}{\partial n}\,d\sigma > \int_C \frac{\partial U_1}{\partial n}\,d\sigma,$$

and

$$f_2(F) > f_1(F).$$

2.1. **Increasing sequences of potentials.** A function which is the limit of an increasing sequence of superharmonic functions and is not identically $+\infty$ is also superharmonic. A similar proposition is the following:

THEOREM. *A function $u(M)$ which is the limit of an increasing sequence of potential functions $U_n(M)$ of positive masses $f_n(e)$ on F, such that $f_n(F)$ is bounded, independently of n, is also a potential of a positive distribution $f(e)$ on F, with*

$$f(F) = \lim_{n=\infty} f_n(F).$$

† An obvious generalization of de la Vallée Poussin's theorem, loc. cit., p. 210.

‡ Where n stands for the direction of the normal to a simple closed surface it is taken as positive towards the interior.

In fact, $u(M)$ is not identically $+\infty$ since the convergence of the sequence is uniform outside a sufficiently large sphere. Hence $u(M)$ is superharmonic. Moreover, as is well known,† there exists (by the Cantor "diagonal process") a subsequence $f_{n_i}(e)$ and a rectangular net R, such that for every mesh ω of R we have

$$f(\omega) = \lim_{n_i=\infty} f_{n_i}(\omega), \tag{6}$$

where $f(e)$ is a certain distribution of positive mass on F. And if $\phi(P)$ is any continuous function,

$$\lim_{n_i=\infty} \int_W \phi df_{n_i} = \int_W \phi df. \tag{7}$$

The equations (6), (7) are characteristic of what Radon calls *weak convergence*.‡

Let $U(M)$ be the potential of $f(e)$. We show that $U(M)\equiv u(M)$, by using the mean value on a spherical surface.

2.2. **Mean value on a spherical surface.** Denote by $A_u(\rho, Q)$ the average value of a function $u(M)$ on a spherical surface $C(\rho, Q)$ of radius ρ and center Q:

$$A_u(\rho, Q) = \frac{1}{4\pi\rho^2}\int_{C(\rho,Q)} u(M)dM. \tag{8}$$

We have, for the potential $U(M)$,

$$\begin{aligned} A_U(\rho, Q) &= \frac{1}{4\pi\rho^2}\int_{C(\rho,Q)} dM \int_W \frac{1}{MP}\, df(e_P) \\ &= \frac{1}{4\pi\rho^2}\int_W df(e_P) \int_{C(\rho,Q)} \frac{1}{MP}\, dM, \\ A_U(\rho, Q) &= \int_W h^{1/\rho}(Q, P)df(e_P) \end{aligned} \tag{8'}$$

from the fact that $\int_C [1/(MP)]dM$ is the potential at P of a uniform distribution on $C(\rho, Q)$. But $h^{1/\rho}(Q, P)$ is a continuous function of P.

For any superharmonic function u, we have

$$u(Q) = \lim_{\rho=0} A_u(\rho, Q). \tag{9}$$

For $u(Q) \leqq \lim\inf\ (\rho=0) A_u(\rho, Q)$, being lower semicontinuous, and $u(Q) \geqq A_u(\rho, Q)$ from the property (ii); that is, $u(Q)$ has what may be called the "super-mean" property. Hence $\lim\ (\rho=0)A_u(\rho, Q)$ exists and satisfies (9).

† J. Radon, *Über lineare Funktionalgleichungen*, Sitzungsberichte der Akademie der Wissenschaften in Wien, vol. 128 (1919), pp. 1083–1121, p. 1092.

‡ Ibid., p. 1088. The equation (7) is, in fact, Radon's definition of weak convergence. For the sake of its more evident relation to the structure of the Stieltjes integral, however, we shall say that the sequence of positive mass distributions $\{f_n(e)\}$ converges weakly to $f(e)$ if $f_n(W)$ is bounded, independently of n, and $\lim\ (n=\infty) f_n(\omega) = f(\omega)$ for every mesh ω of some rectangular net.

Moreover, from (ii), it is evident that if $u(M)$ is superharmonic, $A_u(\rho, Q)$ is monotone increasing as ρ tends to zero. In particular, $U(Q) = \lim\ (\rho = 0)$ $A_U(\rho, Q)$.

Returning to the statement to be proved, we note that from (8), since the U_n form a monotonic sequence,

$$A_u(\rho, Q) = \lim_{n=\infty} A_{U_n}(\rho, Q),$$

and from (8′), by means of the weak convergence property (7),

$$A_U(\rho, Q) = \lim_{n_i=\infty} A_{U_{n_i}}(\rho, Q).$$

Hence $A_u(\rho, Q) = A_U(\rho, Q)$, and accordingly, by (9),

$$u(Q) = U(Q).$$

But this is what was to be proved. If we take the integral over a large spherical surface C, we have $\int_C (du/dn) d\sigma = \lim (n = \infty) \int_C (dU_n/dn) d\sigma$, or $f(F) = \lim (n = \infty) f_n(F)$.

Evidently a sufficient condition that $f_n(F)$ be bounded is that $u(M)$ be not identically infinite. For in that case $u_n(M)$ will converge uniformly to $u(M)$ in any closed region which has no points in common with F.

3. **Generalized derivatives.** We recall some properties of generalized derivatives, with special relation to the potential function.† Since the space integral of $1/(MP^2)$ extended over a bounded domain Ω is convergent,—in fact, for every P,

$$\int_\Omega \frac{1}{MP^2} dM \leqq 4\pi d \tag{10}$$

where d is the diameter of the domain,—it is easily verified that the quantity $U_\alpha(M)$,

$$U_\alpha(M) = \int_W \frac{\cos (MP, \alpha)}{MP^2} df(e_P),$$

α being a fixed direction in space, is a summable function, spatially, of M, and that

$$\int_\Omega dM \int_W \frac{\cos (MP, \alpha)}{MP^2} df(e_P) = \int_W df(e_P) \int_\Omega \frac{\cos (MP, \alpha)}{MP^2} dM. \tag{11}$$

† G. C. Evans, *Complements of potential theory*, II, American Journal of Mathematics, vol. 55 (1933), pp. 29–49.

We denote by $\Phi_\alpha(e)$ the function of point sets generated by $\int_e U_\alpha(M)dM$. *It is absolutely continuous*. Moreover, since wherever $U_x(M)$, $U_y(M)$, $U_z(M)$ exist,

$$(12)\qquad U_\alpha(M) = U_x(M)\cos(x,\alpha) + U_y(M)\cos(y,\alpha) + U_z(M)\cos(z,\alpha),$$

it follows that

$$(13)\qquad \Phi_\alpha(e) = \Phi_x(e)\cos(x,\alpha) + \Phi_y(e)\cos(y,\alpha) + \Phi_z(M)\cos(z,\alpha),$$

for every spatially measurable point set e.

We denote by $D_\alpha U$ *the Lebesgue derivative of the absolutely continuous function of point sets* $\Phi_\alpha(e)$; it exists almost everywhere. In particular, from (13), it exists wherever D_xU, D_yU, D_zU exist, and it satisfies the relations

$$(14)\qquad \begin{aligned} D_\alpha U &= D_xU\cos(x,\alpha) + D_yU\cos(y,\alpha) + D_zU\cos(z,\alpha),\\ D_\alpha U &= U_\alpha, \end{aligned}$$

except possibly on a set of spatial measure zero which is independent of α.

Finally, from the convergence of $\int_\Omega dM\int[1/(MP^2)]df(e_P)$ it follows that $\int[1/(MP^2)]df(e_P)$, $\int[\cos(MP,x)/(MP^2)]df(e_P)$, $\int[1/(MP)]df(e_P)$ represent summable functions of x on almost all lines of direction x. It may be easily verified that on any line l where $\int[1/(MP^2)]df(e_P)$ is summable the total mass must be zero, and that on one of these non-exceptional lines l of direction x, if we select a point A where $U(A)$ is finite,

$$\begin{aligned} \int_A^B dx_M\int_W \frac{\cos(MP,x)}{MP^2}\,df(e_P) &= \int_{W-l} df(e_P)\int_A^B \frac{\cos(MP,x)}{MP^2}\,dx_M\\ &= \int_{W-l}\left[\frac{1}{BP}-\frac{1}{AP}\right]df(e_P) = \int_W\left[\frac{1}{BP}-\frac{1}{AP}\right]df(e_P)\\ &= U(B)-U(A), \end{aligned}$$

so that $U(B)$ exists everywhere on l and is absolutely continuous in x.†

That is to say, *on almost all lines with a given direction* α, $U(M)$ *is absolutely continuous as a function of distance on the line, and its partial derivative exists and has almost everywhere on the line the value*

$$(15)\qquad \frac{\partial U}{\partial\alpha} = D_\alpha U = U_\alpha.$$

That this relation holds almost everywhere in space follows from the spatial

† A generalization of these formulas, for integration along a curve, is given by W. H. Binney, *An elliptic system of integral equations on summable functions*, in the present number of these Transactions, pp. 254–265; see Lemma C.

measurability of the partial derivative numbers. The exceptional set, however, is not shown to be independent of α.

For a surface σ, bounding a domain Ω, sufficiently smooth for Green's theorem to hold, we have

$$(16)\qquad \int_\Omega D_\alpha U dM = \int_\Omega \frac{\partial U}{\partial \alpha} dM = \int_\Omega U_\alpha dM = \int_\sigma U \cos (n, \alpha) dM,$$

which is a sort of three-dimensional statement of absolute continuity along the direction α. The author has used the phrase "U is a potential of its vector or generalized derivative $D_\alpha U$" to signify the relation

$$(16')\qquad \int_\Omega D_\alpha U dM = \int_\sigma U \cos (x, \alpha) dM, \text{ for all } \alpha,$$

even if $\partial U/\partial\alpha$ may fail to exist.

By Riesz's fundamental theorem, it follows that (16) applies to a function u for a bounded region Ω contained with its smooth boundary within any domain in which u is superharmonic.

4. **Further properties of the average. Riesz's theorem.** In order to illustrate further useful properties of the average we shall give a brief proof of the theorem on the resolution of a superharmonic function into a potential and a harmonic function, in line with the ideas of F. Riesz and T. Radó.

Denote by $u(\rho, Q)$ or $a_u(\rho, Q)$ or, for brevity, u_ρ, the average of a function $u(M)$ over a spherical region $\Gamma(\rho, Q)$ of center Q and radius ρ; we take the same ρ for all Q. Similarly, denote by $u(\rho', \rho'', Q)$ or $u_{\rho'\rho''}$ the iterated average, obtained by averaging $u(\rho', M)$ over a sphere of radius ρ'' and center Q. We have

$$(17)\qquad u(\rho, Q) = \frac{3}{\rho^3}\int_0^\rho A_u(\rho, Q)\rho^2 d\rho,$$

A_u being the mean on the spherical surface, as before. Significant properties of these averages are well known and easily established.† If $u(M)$ is summable in a domain Ω, $u(\rho, M)$ is continuous in any portion of Ω distant from the boundary of Ω by as much as ρ, and $\lim\ (\rho=0)\ u(\rho, M) = u(M)$ for almost all M; $\partial u(\rho, M)/\partial x$, $\partial u(\rho, M)/\partial y$, $\partial u(\rho, M)/\partial z$ exist and equal the generalized derivatives almost everywhere; if $u(M)$ is a potential function of its generalized or vector derivative, as in (16′), §3, then

† H. E. Bray, *Proof of a formula for an area*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 264–270; *Green's lemma*, Annals of Mathematics, vol. 26 (1925), pp. 278–286; T. Radó, *Remarques sur les fonctions subharmoniques*, Comptes Rendus de l'Académie des Sciences, vol. 186 (1928), pp. 346–348; F. Riesz, Memoir (2) cited in §2, see p. 342 ff., where other references are given.

$$(18)\qquad a_{D_\alpha u}(\rho, Q) = D_\alpha a_u(\rho, Q) = \frac{\partial u(\rho, Q)}{\partial \alpha},$$

these quantities being continuous. Hence by iterating the averaging operation the derivatives to any order may be made continuous.

If $u(M)$ is superharmonic in Ω, $A_u(\rho, M)$ and $a_u(\rho, M)$ are superharmonic in any portion of Ω distant from the boundary of Ω by more than ρ. In fact, if we write $u(Q)=u(0, 0, 0)$, $u(M)=u(x, y, z)$, $u(P)=u(x+\xi, y+\eta, z+\zeta)$, and formulate explicitly the averages, we see that the successive averaging operations are commutative; that is, for all ρ' sufficiently small

$$(19)\qquad \begin{aligned} u(\rho, \rho', Q) &= \frac{3}{4\pi\rho'^3}\int_{\Gamma(\rho',0)} dx\,dy\,dz\, \frac{3}{4\pi\rho^3}\int_{\Gamma(\rho,0)} u(x+\xi, y+\eta, z+\zeta)d\xi\, d\eta\, d\zeta \\ &= u(\rho', \rho, Q). \end{aligned}$$

Now $u(\rho, M)$ is continuous in M; moreover from (ii), §2,

$$u(\rho, Q) = \frac{3}{4\pi\rho^3}\int_{\Gamma(\rho,Q)} u(M)dM \geqq \frac{3}{4\pi\rho^3}\int_{\Gamma(\rho,Q)} u(\rho', M)dM = u(\rho', \rho, Q),$$

so that, from (19),

$$u(\rho, Q) \geqq u(\rho, \rho', Q) = \frac{3}{4\pi\rho'^3}\int_{\Gamma(\rho',Q)} u(\rho, R)dR$$

which is a substitute for the condition (ii), and makes $u(\rho, M)$ superharmonic. A similar demonstration applies to the A-operation.

Again let $u(M)$ be superharmonic in Ω. The function $u(\rho_1, \rho_2, \cdots, \rho_k, Q)$ is a weighted mean of $u(M)$ over a sphere $\Gamma(\rho_1+\rho_2+\cdots+\rho_k, Q)$, such that $u(\rho_1, \rho_2, \cdots, \rho_k, Q)\leqq u(Q)$. Moreover, since $u(M)$ is lower semicontinuous, $u(Q)\leqq \liminf\,(M=Q)u(M)$; hence

$$\lim u(\rho_1, \rho_2, \cdots, \rho_k, Q) = u(Q),\ Q \text{ in } \Omega,$$

as $\rho_1, \rho_2, \cdots, \rho_k$ tend independently to zero. In particular,

$$(19')\qquad \lim_{\rho=0} u^{(k)}(\rho, Q) = u(Q),\ Q \text{ in } \Omega,$$

where $u^{(k)}(\rho, M)=u(\rho_1, \rho_2, \cdots, \rho_k, M)$ with $\rho_1=\rho_2=\cdots=\rho_k=\rho$, and increases monotonically as ρ decreases to zero.

Let D be a bounded domain, Ω a domain contained with its boundary Ω^* in D. Let Ω_i, $i=1, 2, 3$, be intermediate domains with boundaries Ω_i^* such that we have

$$\Omega+\Omega^* \text{ in } \Omega_1;\ \Omega_i+\Omega_i^* \text{ in } \Omega_{i+1},\ i=1, 2;\ \Omega_3+\Omega_3^* \text{ in } D,$$

and Ω_i^* is sufficiently smooth for application of Green's Theorem to regions bounded internally or externally by it.

THEOREM OF F. RIESZ. *If $u(M)$ is superharmonic in D, it may be written in the form*

$$u(M) = U(M) + v(M), \quad M \text{ in } \Omega,$$

where $U(M)$ is the potential of a distribution of positive mass on Ω, finite in total amount, and $v(M)$ is harmonic in Ω.

Let $u^{(1)}(M)$, $u^{(2)}(M)$, $\cdots$, $u^{(p)}(M)$, $\cdots$ be a monotonic-increasing sequence of continuous superharmonic functions, with limit $u(M)$, in a region Ω_4 which contains $\Omega_3+\Omega_3^*$; for instance, let $u^{(p)}(M)$ be the average $u(1/(p_0+p), M)$, with p_0 fixed and sufficiently great. Define

$$\begin{aligned} u_p(M) &= u^{(p)}(M), \quad M \text{ in } \Omega_1 + \Omega_1^*, \\ &= w_p(M), \quad M \text{ in } \Omega_3 - (\Omega_1 + \Omega_1^*), \\ &= u^{(p)}(M), \quad M \text{ in } \Omega_4 - \Omega_3, \end{aligned}$$

where $w_p(M)$ is the function which is harmonic in $\Omega_3-(\Omega_1+\Omega_1^*)$ and takes on continuously the boundary values $u^{(p)}(M)$ on Ω_1^* and Ω_3^*.

Then $u_p(M)$ is continuous in Ω_4 and evidently possesses the super-mean property. It is therefore superharmonic in Ω_4. Moreover the sequence $u_p(M)$ is monotonic-increasing, and $\leqq u(M)$, converging accordingly to a function $w(M)$, superharmonic in Ω_4. We note that $w(M)$ is harmonic in $\Omega_3-(\Omega_1+\Omega_1^*)$ and identical with $u(M)$ in $\Omega_1+\Omega_1^*$.

We take 4ρ small in comparison with the distances between Ω_3^* and the boundary of Ω_4, between Ω_3^* and Ω_2^*, and between Ω_2^* and Ω_1^*; and consider the function $w(\rho, \rho, \rho, \rho, M) = w^{(4)}(\rho, M)$, which is superharmonic in a region which contains $\Omega_3+\Omega_3^*$ and is identical with $w(M)$ in a neighborhood of Ω_2^*, where it is harmonic. The function $w^{(4)}(\rho, M)$ has continuous third-order partial derivatives, and tends, increasing monotonically, to $w(M)$ in a domain which includes Ω_3, as ρ tends to zero. By means of Green's theorem, as applied to Ω_2, we have the decomposition

$$w^{(4)}(\rho, M) = U_4(M) + v_0(M), \quad M \text{ in } \Omega_2,$$

where $U_4(M)$ is the potential of a distribution of positive continuous density on Ω_2, finite in total amount, and where $v_0(M)$ is harmonic in Ω_2, continuous on $\Omega_2+\Omega_2^*$. Since $v_0(M)$ involves merely the boundary values of $w^{(4)}$ and $dw^{(4)}/dn$ on Ω_2, it follows that $v_0(M)$ will remain independent of ρ, as ρ tends to zero.

The potential $U_4(M)$ therefore increases monotonically as ρ tends to zero. Moreover, since $U_4(M)$ is identical with $w(M)-v_0(M)$ in the neighborhood of Ω_2, and is harmonic there, the total mass is given by 4π times the integral of the normal derivative of $w(M)-v_0(M)$, and does not involve ρ. Hence by the theorem of §2.1, the function

$$U_1(M) = \lim_{\rho=0} U_4(M)$$

is the potential of a distribution of positive mass $f(e)$ on a bounded set, and is bounded in total amount.

We have

$$u(M) = w(M) = v_0(M) + \int_W \frac{1}{MP}\, df(e_P), \qquad M \text{ in } \Omega.$$

Writing

$$f(e) = f(e\cdot\Omega) + f(e\cdot(W - \Omega)),$$

$$U(M) = \int_W \frac{1}{MP}\, df(e_P\cdot\Omega), \qquad v(M) = v_0(M) + \int_W \frac{1}{MP}\, df(e_P\cdot(W - \Omega)),$$

we shall have

$$u(M) = U(M) + v(M), \qquad M \text{ in } \Omega,$$

according to the conditions of the theorem.

II. Limiting values of potential function

In general terms it seems safe to say that the potential of a positive distribution of mass is greater where mass is than where it is not. In fact, the potential cannot take on its upper bound at a point of positive distance from the mass, being harmonic at such a point. Nevertheless it is evident that a potential need not take on its upper bound at any point whatever in space, and if our naive idea is to be made precise, it must involve the limiting values of potential functions as we approach points of the mass distribution.

We note that for any point M whatever, on account of the lower semi-continuity of $U(M)$, and equation (9), §2.1,

$$(1) \qquad U(M) = \liminf_{M'=M} U(M'), \qquad M' \text{ in } W.$$

5. **Points of continuity.** In terms of the notation of §1 we have the following

THEOREM. *Let Q be a point of t, not an isolated point, and $U(Q)$ finite; if*

$$(2) \qquad \lim_{P=Q} U(P) = U(Q), \qquad P \textit{ in } t,$$

then

$$\lim_{M=Q} U(M) = U(Q), \qquad M \text{ in } T. \tag{3}$$

For this theorem, T may be any domain containing no points of F, whose boundary t consists of points of F.

On account of (1) it is sufficient to prove that

$$\limsup_{M=Q} U(M) \leqq U(Q), \qquad M \text{ in } T.$$

We assume that the theorem is false, and that therefore there is a sequence of points $M_1, M_2, \cdots$ in T, and a number $\epsilon > 0$, such that

$$U(M_i) > U(Q) + \epsilon, \qquad \lim_{i=\infty} M_i = Q. \tag{4}$$

Let $\epsilon_1, \epsilon_2, \cdots$ be a sequence of positive numbers such that

$$\epsilon_1 + \epsilon_2 + \cdots < \epsilon/2.$$

Let $\Gamma(\rho, Q)$ be an open spherical neighborhood with center Q and radius ρ, small enough so that the contribution $U_{\Gamma_\rho}(Q)$ to the potential at Q from the mass on $F \cdot \Gamma(\rho, Q)$ satisfies the inequality (see (5) §1)

$$U_{\Gamma_\rho}(Q) < \epsilon_1.$$

It follows therefore that

$$U_{W-\Gamma_\rho}(Q) > U(Q) - \epsilon_1.$$

We suppose also that ρ is small enough so that, for P in $t \cdot \Gamma(\rho, Q)$,

$$U(P) < U(Q) + \epsilon_2.$$

But $U_{W-\Gamma_\rho}(P)$ is continuous at Q for any method of approach; let therefore $\Gamma(\delta, Q)$ be a second spherical neighborhood, concentric with $\Gamma(\rho, Q)$, with $\delta < \rho$, δ being small enough so that, for P in $\Gamma(\delta, Q)$,

$$U_{W-\Gamma_\rho}(P) > U_{W-\Gamma_\rho}(Q) - \epsilon_3 > U(Q) - \epsilon_1 - \epsilon_3.$$

Thus we have

$$\begin{aligned} U_{\Gamma_\rho}(P) &< U(Q) + \epsilon_2 - (U(Q) - \epsilon_1 - \epsilon_3), \\ U_{\Gamma_\rho}(P) &< \epsilon_1 + \epsilon_2 + \epsilon_3, \qquad P \text{ in } t \cdot \Gamma(\delta, Q). \end{aligned} \tag{5}$$

Denote the distance M_iQ by δ_i; we may assume without loss of generality that $\delta_i < \delta/2$ for all i. Let Q_i be a point in F at the minimum distance, say δ_i', from M_i. Such a point Q_i exists, since F is closed; moreover Q_i lies in t, since otherwise the segment Q_iM_i would contain a point of t nearer to M_i

than Q_i. Further, Q_i lies in $\Gamma(\delta, Q)$, since $QQ_i \leqq QM_i + M_iQ_i < \delta/2 + \delta/2$. Finally, $\lim (i = \infty) Q_i = Q$.

We have

$$Q_iP \leqq Q_iM_i + M_iP \leqq 2M_iP, \qquad P \text{ in } F \cdot \Gamma(\rho, Q). \tag{6}$$

In fact, $Q_iM_i = \delta_i'$, $M_iP \geqq \delta_i'$. Also, for P in $F - F \cdot \Gamma(\rho, Q)$,

$$Q_iP \leqq Q_iM_i + M_iP = \left(1 + \frac{Q_iM_i}{M_iP}\right) M_iP$$

$$\leqq \left(1 + \frac{\delta_i'}{\rho - \delta_i}\right) M_iP,$$

since for such P, $M_iP \geqq QP - QM_i \geqq \rho - \delta_i$. Accordingly, since $\rho > \delta > \delta_i \geqq \delta_i'$,

$$Q_iP < \left(1 + \frac{\delta_i}{\rho - \delta}\right) M_iP, \qquad P \text{ in } F - F \cdot \Gamma(\rho, Q). \tag{7}$$

Now, from (6), (7), making use of the properties (C), (P) of the general integral,

$$U(M_i) = U_{\Gamma_\rho}(M_i) + U_{W-\Gamma_\rho}(M_i) = \int_W \frac{1}{M_iP}\, df(e_P \cdot [F \cdot \Gamma(\rho, Q)])$$

$$+ \int_W \frac{1}{M_iP}\, df(e_P \cdot [F - F \cdot \Gamma(\rho, Q)])$$

$$< 2U_{\Gamma_\rho}(Q_i) + \left(1 + \frac{\delta_i}{\rho - \delta}\right) U_{W-\Gamma_\rho}(Q_i)$$

$$= U_{\Gamma_\rho}(Q_i) + U(Q_i) + \frac{\delta_i}{\rho - \delta}\, U_{W-\Gamma_\rho}(Q_i)$$

$$< \epsilon_1 + \epsilon_2 + \epsilon_3 + U(Q) + \epsilon_2 + \frac{\delta_i}{\rho - \delta}\,(U(Q) + \epsilon_2).$$

Choose now i great enough so that

$$\frac{\delta_i}{\rho - \delta}\,(U(Q) + \epsilon_2) < \epsilon_4.$$

Then

$$U(M_i) < U(Q) + \epsilon_1 + 2\epsilon_2 + \epsilon_3 + \epsilon_4 < U(Q) + \epsilon.$$

But this statement contradicts (4). Accordingly the proof is complete.

Corollary. *Let Q be a point of t, isolated or not, and $U(Q)=+\infty$. Then*

$$\lim_{M=Q} U(M) = U(Q), \qquad M \text{ in } W.$$

In fact, this is equation (1) in this case.

5.1. **Frequency of points of continuity.** Let the set t be without isolated points; it is then perfect. Let Γ_1 be any closed spherical region which contains F in its interior. The functions

$$U^{(N)}(P) = \int_W h^N(P, P')df(e_{P'})$$

are, on Γ_1, positive, continuous, and bounded away from zero by a lower bound independent of N, as N tends to $+\infty$. Accordingly the functions

$$v_N(P) = 1/U^{(N)}(P)$$

are continuous and uniformly bounded on Γ_1, and the function

$$v(P) = 1/U(P)$$

is a function of the *first class of Baire* on Γ_1. It is therefore *punctually discontinuous* on Γ_1.† Given any perfect subset E of t (for instance, all the points of t in a closed spherical neighborhood of a point P of t) there will be points of E at which $v(P)$ is continuous, considered only on E. In other words, admitting $+\infty$ as a possible value of U, we have the following proposition:

Theorem. *If t is perfect, and P is given in t, then in any neighborhood of P there is a point Q of t such that*

$$\lim_{P'=Q} U(P') = U(Q), \qquad P' \text{ in } t.$$

By an application of the theorem of §5 we have the following corollary:

Corollary I. *If t is perfect, and P is given in t, then in any neighborhood of P there is a point Q of t such that*

$$\lim_{M=Q} U(M) = U(Q), \qquad M \text{ in } T+t.$$

The following corollary may also be stated, and proved in the same manner as the above. In fact, in the theorem of §5 we may replace t by F, and T by CF, although CF is not necessarily a domain, Q being a frontier point of CF.

† Lebesgue, *Leçons sur l'Intégration*, Paris, 1928, p. 203.

COROLLARY II. *If F is perfect and P is a point of F, there will be, in any neighborhood of P, a point Q of F such that*

$$\lim_{M=Q} U(M) = U(Q), \qquad M \text{ in } W. \tag{8}$$

This corollary affords an immediate proof of Kellogg's Lemma (see §18).

6. **The superior limit of the potential function.** We return to the set t as in §5, which is closed but not necessarily perfect, and let T be a bounded or an unbounded domain.

THEOREM.† *Let Q be a point of t, not an isolated point, and let k, assumed to be finite, be the superior limit of $U(P)$, P in t, as P tends to Q. Given $\epsilon > 0$, we can find Q_1 in t, arbitrarily close to Q, so that*

$$\limsup_{M=Q_1} U(M) < U(Q_1) + \epsilon < k + 2\epsilon, \qquad M \text{ in } T. \tag{9}$$

With $\epsilon_1, \epsilon_2, \cdots$ as before, we choose a neighborhood Ω of Q of diameter small enough so that $U(P) < k + \epsilon_2$ for P in $t \cdot \Omega$. For Q_1 we take any point in $t \cdot \Omega$ for which $U(Q_1) > k - \epsilon_5$; this choice of Q_1 is possible by definition of k. The proof of the theorem from this point on is substantially that of the theorem of §5, with Q of §5 replaced by Q_1. Hence it need not be repeated here.

The following simple remarks supplement the theorem just given.

I. *If Q is an isolated point of t, either $U(M)$ is continuous (and harmonic) at Q, or else* $\lim (M=Q) U(M) = U(Q) = +\infty$, *for M in W.*

If there is no mass on Q, $U(M)$ is bounded and harmonic in the neighborhood of Q and continuous at Q, therefore harmonic at Q. If there is a point mass at Q, $U(Q) = +\infty$, $\lim (M=Q)\, U(M) = +\infty$.

II. *Let Σ be any domain and s its boundary, Q any point of s. Then*

$$\limsup_{P=Q} U(P) \leqq \limsup_{M=Q} U(M), \qquad P, Q \text{ in } s,\ M \text{ in } \Sigma. \tag{10}$$

For there exists a sequence of points P_i of s, tending to Q, such that $\lim U(P_i)$ exists and equals $\lim\sup (P=Q) U(P)$. But there is a sequence of points M_i in Σ, $M_iP_i < \epsilon/2^i$, $U(M_i) > U(P_i) - \epsilon/2^i$, ϵ given >0; for $U(M)$ is lower semicontinuous. Hence, admitting the value $+\infty$ as a possible limit, the equation (10) is established.

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† The theorem is slightly sharper than the lemma of §2 of G. C. Evans, *Application of Poincaré's sweeping-out process*, Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 457–461, but the proof is quite similar. In the cited lemma t was assumed to contain all the mass, instead of being merely a frontier of the mass, as here.

The theorem of §4 is a corollary of the above, by taking $Q_1 = Q$.

III. We leave unanswered the question†

$$\text{u.b. } U(P)(P \text{ in } t) = \text{u.b. } U(M)(M \text{ in } T)? \tag{10'}$$

7. **Inferior limit of the potential function.** We shall see in §7.2 that there may exist points of the mass where the potential is actually less than its lower limit for approach not on the mass. For their consideration there is no gain in restricting the sets in which they lie to points of the mass, specifically, and accordingly we discuss them with reference to the sets s, Σ, B of §1. We are thus dealing with the properties of potentials as positive superharmonic functions rather than as explicitly given integrals.

If there exists a potential $U(M)$ of positive mass on F such that

$$U(Q) < \liminf_{M=Q} U(M), \qquad Q \text{ in } s, \ M \text{ in } \Sigma, \tag{11}$$

we say that Q is an *exceptional point of s with respect to* Σ; similarly we speak of exceptional points of s with respect to $\Sigma+B$, etc. As is evident from the definition, we are dealing with geometrical properties of the sets in question.

We use the symbol $C(\rho, Q, E)$ for the portion of the spherical surface $C(\rho, Q)$ which is common to it and a set E, measurable Borel. Such a portion is also measurable Borel.

7.1. **Special cases.** We prove the following theorem:

THEOREM I. *If Q is an exceptional point of s with respect to Σ,*

$$\lim_{\rho=0} \frac{C(\rho, Q, \Sigma)}{C(\rho, Q, s + B)} = 0;$$

also, if Q is an exceptional point of s with respect to $\Sigma+B$,

$$\lim_{\rho=0} \frac{C(\rho, Q. \Sigma + B)}{C(\rho, Q, s)} = 0.$$

Suppose that

$$\liminf_{M=Q} U(M) = U(Q) + h, \qquad M \text{ in } \Sigma, \ h > 0.$$

Let θ be any fixed number $0<\theta<1$. Since $U(M)$ is lower semicontinuous, it follows that, given $\epsilon>0$, there is a spherical neighborhood $\Gamma(\rho_1, Q)$, such that

$$U(M) > U(Q) + \theta h, \qquad M \text{ in } \Sigma, \ QM < \rho_1,$$
$$U(P) > U(Q) - \epsilon, \qquad P \text{ in } W, \ QP < \rho_1.$$

Hence

† The question is now answered in the affirmative, by A. J. Maria, *The potential of a positive mass and the weight function of Wiener*, Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 485–489.

$$C(\rho, Q)U(Q) > C(\rho, Q, \Sigma)[U(Q) + \theta h] + C(\rho, Q, s + B)[U(Q) - \epsilon],$$
$$0 > \theta h C(\rho, Q, \Sigma) - \epsilon C(\rho, Q, s + B),$$

(12)
$$\frac{C(\rho, Q\, \Sigma)}{C(\rho, Q, s + B)} < \frac{\epsilon}{\theta h}, \qquad \rho < \rho_1.$$

Similarly, if

$$\liminf_{M=Q} U(M) = U(Q) + h, \qquad M \text{ in } \Sigma + B,\ h > 0,$$

there exists ρ_2 such that

(13)
$$\frac{C(\rho, Q, \Sigma + B)}{C(\rho, Q, s)} < \frac{\epsilon}{\theta h}, \qquad \rho < \rho_2.$$

From this the conclusions of the theorem are evident.

COROLLARY. *An exceptional point of s with respect to Σ is a point of s of spatial density unity in s+B. If s is a set of Lebesgue spatial measure zero it has no exceptional points with respect to Σ+B.*

We remark that the exceptional points of s with respect to Σ must all be regular boundary points of Σ, since an irregular boundary point of Σ cannot be a point of spatial density unity on $s+B$ (see §23, below).

Perfect totally disconnected sets, of which the typical example is the spatial discontinuum, need not be of zero spatial measure. But they are closed sets of *dimension zero* in the sense of Menger. The following theorem about such sets may be proved very simply.

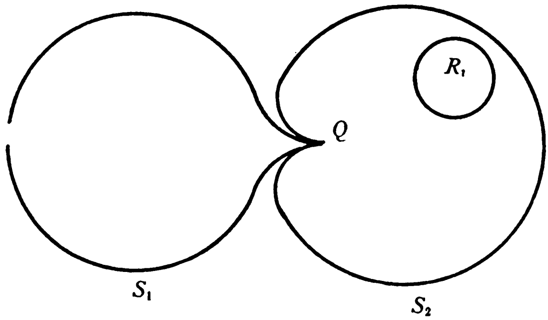
THEOREM II. *If s is of dimension zero at Q, Q cannot be an exceptional point with respect to Σ. If s is of dimension zero, none of its points are exceptional with respect to Σ.*

If s is of dimension zero at Q, in any neighborhood about Q there is contained a neighborhood Ω, to which also Q belongs, whose boundary Ω^* contains no points of s. The set Ω^* is at a positive distance from the closed set s, and, for Ω small enough, since it contains at least one point of Σ, lies entirely in Σ. The lower bound of $U(M)$ for Ω is taken on at some point M_1 of Ω^*. Hence by taking a sequence of neighborhoods with diameters which approach zero as a limit we obtain a sequence of points $M_1, M_2, \cdots$, $\lim M_i = Q$, M_i in Σ, such that $U(Q) \geqq U(M_i)$.

The second part of the theorem is a consequence of the first, for a set of dimension zero is of dimension zero at every one of its points.

7.2. **An exceptional case.** The accompanying figure illustrates a point Q of s for which lim inf $(M=Q)$ $U(M)>U(Q)$, M in Σ. The set B is vacuous, and $F\equiv s$, mass being deposited only on s.

We construct a surface of revolution S_1, made in part of a sphere with center O and in part of an exterior Lebesgue spine with vertex at Q, the sphere being pierced opposite Q on QO, so that the complement of S_1 becomes a domain. To S_1 is applied a conductor potential distribution (see §15) with



potential $U_1(M)$. The point Q being irregular for the domain $W-S_1$, we know, from the symmetry of the figure about OQ, that, as M approaches Q along the extension of OQ (from the right, in the figure),

$$\lim_{M=Q} U_1(M) = U_1(Q) = 1 - k < 1.$$

We now adjoin to the end of the spine, on the same axis of revolution, a closed surface of revolution S_2, exterior to S_1, such that on S_2, in the neighborhood of Q, $U_1(M)\geqq 1-k/2$. Throughout B_2, the region interior to S_2, we distribute a uniform mass, of density sufficiently small so that its potential nowhere exceeds the value $k/8$. Accordingly the inferior limit of the potential of the total mass, for approach to Q, is still obtained by a path through the region B_2. In the figure so far constructed we have $F=B_2+S_1+S_2$.

We may now eliminate the interior mass and still retain the desired property. Denote by σ the projection of B_2 on the x, y plane, and let the points R_k of σ, for which x and y are both rational, be put in countable order. With R_k as center of base, construct a cylindrical surface σ_k of length l, long in comparison with the diameter of S_2, remove the mass from the interior of the tube, and place a uniform mass on the surface of the tube, so that

(i) the potential of this mass at any point in the set O_k common to the interior of the tube σ_k and to B_2 shall be $\geqq 2$,

(ii) the potential of this mass at Q shall be $<2^{-i}(k/8)$,

(iii) the tube σ_k shall be exterior to the tubes $\sigma_1, \sigma_2, \cdots, \sigma_{k-1}$, if R_k is exterior to them; otherwise R_k shall be omitted from the sequence,

(iv) the sum of the masses on all the σ_k shall be finite. This construction is possible, since a segment of a straight line is a point set of zero capacity in three dimensions; that is, any positive charge on the line will make the potential infinite at some point.

The set $S_1+S_2+B-\Sigma O_k+F'$, where F' denotes the closed cover of $\Sigma\sigma_k$, is closed and bounded. We take this set as the set s, and denote by $U(M)$ the potential of the masses distributed on s. For Σ we have the entire set complementary to s. We have evidently

$$U(Q) \leqq 1 - 3k/4,$$

$$\liminf_{M=Q} U(M) > 1 - k/2, \qquad M \text{ in } \Sigma.$$

Hence

$$\liminf_{M=Q} U(M) > U(Q) + k/4, \qquad M \text{ in } \Sigma.$$

Incidentally, we know that s is of positive spatial measure, of set density 1 at Q, and that Q is a regular boundary point of Σ.

7.3. **A theorem on the inferior limit.** We prove the following

THEOREM. *Let Q be a point of s, x, y two arbitrary directions at right angles, Ω an arbitrary neighborhood of Q. Then Q is not an exceptional point of s with respect to $\Sigma+B$ unless the set $\Omega\cdot s$ contains a family of closed rectangular contours, with directions parallel to x, y, whose vertices constitute a set of positive spatial measure.*

Let z be a direction normal to x, y. Denote by $\Gamma(\rho, Q, E)$ the portion of $\Gamma(\rho, Q)$ which lies in the Borel measurable set E. Then

$$\Gamma(\rho, Q, E) = \int_0^\rho C(\rho, Q, E)d\rho. \tag{14}$$

Suppose the theorem is not true, and thus that

$$\liminf_{M=Q} U(M) = U(Q) + 3h, \qquad M \text{ in } \Sigma + B,\ h > 0. \tag{15}$$

Let E denote the set of points in W where $U(M) \leqq U(Q)+h$; it is closed.

Given $\eta>0$, there exists $\rho_0>0$ so that for $\rho\leqq\rho_0$, $\Gamma(\rho, Q, E)$ is contained in Ω and

$$\begin{aligned}
&\text{meas } \Gamma(\rho, Q, E) \geqq (1 - \eta)4\pi\rho^3/3,\\
&\Gamma(\rho, Q, E) \text{ lies in } s,\\
&\liminf_{M=P} U(M) \geqq U(P) + h, \qquad M \text{ in } \Sigma + B,\ P \text{ in } \Gamma(\rho, Q, E).
\end{aligned} \tag{16}$$

In fact, by (15), for ρ sufficiently small, $U(M) > U(Q) + 2h$, M in $\Sigma + B$, $QM < \rho$. Hence $\Gamma(\rho, Q, E)$ lies in s, and the second and third of conditions (16) are satisfied, for such ρ. Now, similarly to (13), for ρ sufficiently small, given $\epsilon > 0$,

$$C(\rho, Q, W - E) < \frac{\epsilon}{h} C(\rho, Q, E),$$

$$C(\rho, Q) < \frac{\epsilon + h}{h} C(\rho, Q, E),$$

$$\Gamma(\rho, Q) < \frac{\epsilon + h}{h} \Gamma(\rho, Q, E),$$

from which follows the first of equations (16).

On almost any plane $z = \text{const.}$, $U(M)$ is continuous along almost all lines $x = \text{const.}$ and $y = \text{const.}$ (see §3). Select then a non-exceptional plane $z = z_0$, which cuts $\Gamma(\rho_0, Q, E)$ in a set of positive superficial measure, and in this plane a line $y = y_0$, which cuts $\Gamma(\rho_0, Q, E)$ in a set E_x of positive linear measure. On almost all lines $x = \text{const.}$, which pass through points of E_x, $U(M)$ is continuous.

There is a closed subset F_x of E_x of positive measure, such that $\lim (y = y_0)\ U(x, y, z_0) = U(x, y_0, z_0)$ uniformly, for x in F_x; that is, $\delta' > 0$ exists so that

$$(17) \qquad U(x, y, z_0) - U(x, y_0, z_0) < h/4, \qquad |y - y_0| \leqq \delta', \qquad x \text{ in } F_x.$$

In fact, let $\{y_i\}$ be a sequence of values, tending to y_0. Since $U(x, y, z_0)$ is lower semicontinuous, the sets of points in a rectangle $a < x < b$, $y_0 \lesssim y \lesssim y_i$, contained in $\Gamma(\rho, Q)$, where $U(x, y, z_0) > c$ and where $U(x, y, z_0) < c$ respectively, for any c, are measurable Borel; hence their projections on $y = y_0$ are also measurable Borel. But these are respectively the sets where $f_i(x) > c$ and $\phi_i(x) < c$, $f_i(x)$ being the upper bound and $\phi_i(x)$ the lower bound of $U(x, y, z_0)$ considered as a function of y, $y_0 \lesssim y \lesssim y_i$, for x in the interval $a < x < b$. Hence $f_i(x)$ and $\phi_i(x)$ are measurable Borel, and

$$\lim_{i=\infty} f_i(x) = U(x, y_0, z_0), \qquad \lim_{i=\infty} \phi_i(x) = U(x, y_0, z_0), \qquad x \text{ in } E_x.$$

But, by Egoroff's theorem, corresponding to the sequence $\{i\}$, there is a closed subset F_x of E_x, of measure differing arbitrarily little from that of E_x, such that the approach of $f_i(x)$ and $\phi_i(x)$ to their limiting values is uniform, for x in F_x. Also, for $y_0 \lesssim y \lesssim y_i$,

$$\phi_i(x) - U(x, y_0, z_0) \leqq U(x, y, z_0) - U(x, y_0, z_0) \leqq f_i(x) - U(x, y_0, z_0),$$

so that the approach of the middle member of this inequality to zero is uniform for x in F_x.

For almost all the points of F_x the linear set density of F_x is unity. Select one of these non-exceptional points (x_0, y_0, z_0). For almost all lines $y = \text{const.}$ in the plane $z = z_0$, $U(M)$ is continuous. We may therefore apply the sort of argument just used in the case of F_x to the neighborhood of y_0 on the line $x = x_0$. There exists then a set F_y in y, $|y - y_0| \leqq \delta'$, of positive measure on the line $x = x_0$, and a $\delta'' > 0$ such that

$$(18)\quad U(x, y, z_0) - U(x_0, y, z_0) < h/4, \qquad \text{for } |x - x_0| < \delta'', \qquad y \text{ in } F_y.$$

We may choose the (x_0, y_0, z_0), δ', δ'' so that the entire figure lies in the sphere of center Q and radius ρ_0.

Consider then a rectangle in $z = z_0$, composed of two lines $x = x_1$, $x = x_2$, x_1, x_2 in F_x and two lines $y = y_1$, $y = y_2$, y_1, y_2 in F_y, with $|x_1 - x_0|$, $|x_2 - x_0| < \delta''$, and $|y_1 - y_0|$, $|y_2 - y_0| < \delta'$. This rectangular contour lies entirely in s. In fact, for any point on a side parallel to x, say (x, y_1, z_0), we have

$$\begin{aligned} U(x, y_1, z_0) &- U(x_0, y_0, z_0) \\ &\leqq U(x, y_1, z_0) - U(x_0, y_1, z_0) + U(x_0, y_1, z_0) - U(x_0, y_0, z_0) \\ &< h/4 + h/4 = h/2, \end{aligned}$$

by (17), (18). But by the third of equations (16), since $P = (x_0, y_0, z_0)$ lies in $\Gamma(\rho_0, Q, E)$, $M = (x, y_1, z_0)$ cannot lie in $\Sigma + B$. Also for any point on a side parallel to y, say (x_1, y, z_0), we have

$$U(x_1, y, z_0) - U(x_1, y_0, z_0) < h/4.$$

But (x_1, y_0, z_0) lies in $\Gamma(\rho_0, Q, E)$; hence (x_1, y, z_0) cannot lie in $\Sigma + B$.

The set of points (x_1, y_1, z), (x_2, y_2, z), (x_1, y_2, z), (x_2, y_1, z), vertices of rectangular contours which lie in s, for which $|x_2 - x_1| \geqq a$, $|y_2 - y_1| \geqq b$, is closed. Hence the set of vertices of all the contours of the theorem, for given directions x, y, is measurable Borel. But, as we have proved, this set cannot be of zero measure spatially. This completes the theorem.

It will be noticed that Theorem I of §7.1 relates to measure and Theorem II of §7.1 is topological in character. The present theorem partakes of both characters. A set which has an exceptional point, by this theorem, cannot be of zero spatial measure, since it must contain a set of vertices of positive measure; it cannot be of dimension zero, since the neighborhood of an exceptional point must contain contours of dimension one.

In the plane, and for logarithmic potential, a boundary set s which contains an exceptional point with respect to the complement $\Sigma + B$ of s contains a family of rectangular contours, with sides parallel to given orthogonal directions, whose *vertices constitute a set of positive superficial measure.* It follows then that *if s is a boundary set and occludes no points from infinity it has no exceptional points.* This is a topological theorem of which the analogue in three dimensions, as illustrated in §7.2, is not valid.

III. The energy equation

8. The Dirichlet integral and the averaging process. When U is a potential, the quantity

$$(1) \qquad (\nabla U)^2 = \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2$$

has meaning almost everywhere and is a measurable function spatially; for the separate partial derivatives, according to §3, possess these properties. Moreover the quantity in (1) is identical almost everywhere with the expression in terms of generalized or vector derivatives

$$(D_x U)^2 + (D_y U)^2 + (D_z U)^2$$

which is invariant of an orthogonal transformation at all points where $D_x U$, $D_y U$, $D_z U$ exist. We shall discuss the convergence of the Dirichlet integrals

$$(2) \qquad D = D(U) = \int_W (\nabla U)^2 dM,$$

$$(3) \quad D(U, V) = \int_W \left\{ \frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z} \right\} dM = \int_W \nabla(U, V) dM.$$

It is immediately verified that if $\delta(m)$ is a function such that

$$\int_E \phi^2 dM \leqq \delta(m), \qquad \int_E \psi^2 dm \leqq \delta(m)$$

when meas $E \leqq m$, then also, using the notation of §4 for the spatial average,

$$(4) \qquad \left| \int_E \phi(\rho', M)\psi(\rho'', M) dM \right| \leqq \delta(m).$$

We state this fact as a lemma.

Lemma. *Let $\phi(M)$, $\psi(M)$ be summable with their squares over W. The absolute continuity of the integrals over measurable sets E,*

$$\int_E \{\phi(\rho', M)\}^2 dM, \qquad \int_E \phi(\rho', M)\psi(\rho'', M) dM,$$

is uniform as ρ' and ρ'' tend independently to zero.

If $U(M)$ is a potential of a distribution of positive mass on F, we have

$$U(\rho, M) = \int_W k_\rho(M, P) df(e_P) \tag{5}$$

where

$$k_\rho(M, P) = \frac{3\rho^2 - MP^2}{2\rho^3}, \qquad MP < \rho,$$

$$= \frac{1}{MP}, \qquad MP \geqq \rho,$$

is continuous in M, P with continuous partial derivatives of the first order, and is superharmonic as a function of M. Moreover, since $U(\rho, M)$ vanishes continuously at ∞ and is harmonic outside a set F_ρ, of which no point is distant from F by more than ρ, it follows by §2 that $U(\rho, M)$ is itself the potential of a distribution of positive mass $f(\rho, e)$ on F_ρ; and, as is seen by integration over a sufficiently large spherical surface, the total mass is the same as that for $U(M)$. It has already been pointed out, in the proof of the theorem of §2.1 (the $U(\rho, M)$ forming an increasing sequence of potentials), that $f(\rho, e)$ converges weakly to $f(e)$ on a subsequence of values of ρ.

Since $U(M)$ has summable first partial derivatives, and satisfies (16), §3, $U(\rho, M)$ has continuous first partial derivatives and satisfies (18), §4. In particular, $D(U_\rho)$ exists. If $V(M)$ is a second potential of the same kind, $D(U_{\rho'}, V_{\rho''})$ also exists.

9. **Convergence of the Dirichlet integrals.** We shall prove the following

THEOREM. *A necessary and sufficient condition that $D(U)$ converge is that*

$$D(U_\rho) = \int_W \{\nabla U(\rho, M)\}^2 dM$$

remain bounded as ρ tends to zero. Further

$$D(U) = \lim_{\rho=0} D(U_\rho) \tag{6}$$

if $D(U)$ converges.

That the condition is necessary comes immediately from the lemma of §8, applied to $\partial U_\rho/\partial x$, $\partial U_\rho/\partial y$, $\partial U_\rho/\partial z$ separately. For it follows that the absolute continuity of $\int\{\nabla U(\rho, M)\}^2 dM$ is uniform as ρ tends to zero. Hence this integral is bounded, independently of ρ, on any bounded region. But outside a certain bounded region, sufficiently large, $U(\rho, M)$ is identical with $U(M)$, for all ρ, as ρ tends to zero. Hence $\int_W\{\nabla U(\rho, M)\}^2 dM$ is bounded, independently of ρ. Also equation (6) is verified. For since lim $(\rho=0)$ $\{\nabla U(\rho, M)\}^2=\{\nabla U(M)\}^2$ for almost all M, and the absolute continuity of the integral is uniform, it follows by Vitali's theorem that

$$\lim_{\rho=0}\int_D \{\nabla U(\rho, M)\}^2 dM = \int_D \{\nabla U(M)\}^2 dM,$$

where D is any bounded region; on the other hand, the integrals extended over the portion $W-D$ of W, where D is taken sufficiently large, are identical.

The condition is also sufficient. In fact, it is well known and immediate that if $\phi_1(M)$, $\phi_2(M)$, $\cdots$ form a sequence of not negative summable functions, which have a limit, lim $(i=\infty)\phi_i(M)=\phi(M)$ almost everywhere on a perfect set E, then

$$\liminf_{i=\infty}\int_E \phi_i(M)dM \geqq \int_E \phi(M)dM, \tag{7}$$

admitting $+\infty$ as a possible value of the right hand member. Hence if K exists such that

$$\int_W \{\nabla U(\rho, M)\}^2 dM \leqq K, \text{ for all } \rho,$$

it follows from (7), taking $\rho=1/i$, that $\{\nabla U(M)\}^2$ is summable over every bounded region D, and

$$\int_D \{\nabla U(M)\}^2 dM \leqq K.$$

Hence $\{\nabla U(M)\}^2$ is summable over W.

We have also immediately the following corollary:

COROLLARY. *If $U(M)$, $V(M)$ are two potentials of positive masses on F, finite in total amount, such that $D(U)$ and $D(V)$ converge, the integral $D(U, V)$ also converges, and*

$$D(U, V) = \lim D(U_{\rho'}, V_{\rho''}), \tag{8}$$

as ρ' and ρ'' tend independently to zero.

The convergence comes immediately from the inequality $2|\nabla(U, V)| \leqq (\nabla U)^2+(\nabla V)^2$; the limit property is a consequence of the uniform absolute continuity, as in the proof of the theorem.

10. **The energy equation.** We prove the following

THEOREM. *If $D(U)$ exists, then*

$$D(U) = 4\pi \int_W U(P) df(e_P), \tag{9}$$

and if also $D(V)$ exists, and $\mu(e)$ is the mass function for V, then

$$D(U, V) = 4\pi \int_W U(P) d\mu(e_P). \tag{10}$$

Consider first (10). We remember that $U(\rho', M)$, $V(\rho'', M)$ are potentials of positive mass distributions on bounded sets, and prove the following lemma.

LEMMA. *If $D(U_{\rho'}, V_{\rho''}) = 4\pi \int_W U(\rho', P) d\mu(\rho'', e_P)$ and $D(U)$, $D(V)$ exist, then $D(U, V) = 4\pi \int_W U(P) d\mu(e_P)$.*

In fact, by the corollary of §9,

$$D(U, V) = \lim_{\rho'=0} \left\{ \lim_{\rho''=0} D(U_{\rho'}, V_{\rho''}) \right\} = 4\pi \lim_{\rho'=0} \left\{ \lim_{\rho''=0} \int_W U(\rho', P) d\mu(\rho'', e_P) \right\}$$

$$= 4\pi \lim_{\rho'=0} \int_W U(\rho', P) d\mu(e_P)$$

by the weak convergence of $\mu(\rho'', e)$ to $\mu(e)$, over the proper sequence of values of ρ'', the function $U(\rho', M)$ being continuous. Whence, since $U(\rho', P)$ increases monotonically to $U(P)$, as ρ' tends to zero, by the definition of generalized integral,

$$D(U, V) = 4\pi \int_W U(P) d\mu(e_P),$$

which proves the lemma.

To return to the theorem, we note that $U(\rho_1', \rho_2', \rho_3', M)$, $V(\rho_1'', \rho_2'', \rho_3'', M)$ have continuous third partial derivatives, and the identity (10) applies to them as an immediate consequence of Green's theorem, their first derivatives vanishing continuously at infinity like $1/r^2$. Hence by successive applications of the lemma, the equation (10) is established for U, V. Equation (9) is obtained from (10) by writing $V = U$.

COROLLARY I. *If $D(U)$, $D(V)$ are finite, then also $D(U, V)$ is finite and positive.*

In fact, $D(U, V)$ is given by (10).

COROLLARY II. *If $D(U_1)$, $D(U_2)$, $D(V)$ are finite and $U_2 \geqq U_1$ but $U_2 \not\equiv U_1$, then $D(U_2) > D(U_1)$; $D(U_2, V) \geqq D(U_1, V)$.*

Since there is a point where $U_2 > U_1$, there will be a neighborhood of the point in which, almost everywhere, $U_2 > U_1$, by (19′), §4. But U_1, U_2 vanish continuously at ∞. Hence $D(U_2 - U_1) > 0$.

The second inequality of the theorem is an immediate consequence of (10). Moreover, since $D(U_1)$ and $D(U_2)$ are convergent Lebesgue integrals, the same is true of $D(U_2, U_1)$ and of $D(U_2 - U_1, U_1) = D(U_2, U_1) - D(U_1)$, although $U_2 - U_1$ is not necessarily a potential of positive mass; hence also $D(U_2 - U_1, U_2)$ and $D(U_2 - U_1) = D(U_2 - U_1, U_2) - D(U_2 - U_1, U_1)$ are convergent Lebesgue integrals. But

$$D(U_2) = D(U_1 + U_2 - U_1) = D(U_1) + 2D(U_2 - U_1, U_1) + D(U_2 - U_1)$$

where $D(U_2 - U_1) > 0$, and by equation (10), $D(U_2 - U_1, U_1) \geqq 0$.

Corollary III. *The integrals $D(U_{\rho'}, V_{\rho''})$, $D(U_{\rho'})$ are monotonic increasing as ρ', ρ'' decrease independently to zero.*

Corollary IV. *A necessary and sufficient condition that $D(U)$ exist is that the generalized integral $\int_W U(P)df(e_P)$ converge.*

The necessity has already been demonstrated in the theorem. For the sufficiency, we note first that

$$D(U_{\rho'}, U_{\rho''}) = 4\pi \int_W U(\rho', P)df(\rho'', e_P).$$

Choose now a sequence of values ρ_i'', decreasing to zero, such that the corresponding mass distributions $f(\rho_i'', e)$ of the $U(\rho_i'', M)$ converge in the weak sense to $f(e)$. Since $U(\rho', M)$ is continuous,

$$\lim_{\rho_i''=0} D(U_{\rho'}, U_{\rho_i''}) = 4\pi \int_W U(\rho', P)df(e_P).$$

But by Corollary III, $D(U_{\rho'}, U_{\rho''})$ increases monotonically as ρ'' decreases to zero; accordingly

$$D(U_{\rho'}, U_{\rho''}) \leqq 4\pi \int_W U(\rho', P)df(e_P) \leqq 4\pi \int_W U df, \qquad \text{for all } \rho', \rho''.$$

Hence, by taking $\rho'' = \rho$, $\rho' = \rho$, $D(U_\rho)$ is bounded independently of ρ, and $D(U)$ is finite, by the theorem of §9.

Corollary V. *A sufficient condition that $D(U)$ exist is that $U(M)$ be bounded in W.*

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